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# Casimir effect for moving bodies

Giorgio Calucci

Dipartimento di Fisica Teorica dell'Università di Trieste, Trieste I-34014, INFN, Sezione di Trieste, Italy

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**Abstract.** The usual presentation of the Casimir effect refers to the presence of forces between uncharged macroscopic bodies due to the vacuum fluctuations. If the macroscopic bodies are put in relative motion, the boundary conditions are continuously changed and this should lead to an emission of quanta out of the vacuum. The rate of emission is estimated in the simplest possible geometrical and kinematical situations; the effect is found to be easily calculable but very small because the macroscopic bodies are always extremely slow with respect to the speed of light. It is, however, possible that a resonant effect might enhance the process.

## 1. Introduction

The Casimir effect shows the appealing feature of relating forces acting on macroscopic bodies to typical features of quantum field theory [1]. Although this point of view can be an oversimplification, because the microscopic structure of the conductors is essential in establishing the boundary condition for the EM field, it can be kept at least for simple configurations and for low frequencies.

Within this description we can also study a complementary aspect, i.e. the effect of a macroscopic motion on the quantum state. To be definite we can consider a plane capacitor with zero-point EM field inside and then let one of the plates be moved with respect to the other and inquire how the old vacuum is seen in this new condition. Since there will be some mismatch between the vacua some photons will be found. The macroscopic motion is certainly extremely low with respect to the speed of light, which is the typical speed of the quantum system, so the adiabatic approximation should be fully justified and effective.

The EM field shows some complications due to the gauge and polarization degrees of freedom; it may be useful, therefore, to start with a simplified model where these additional aspects are absent and the space is 1D and then turn to the real problem. Finally, a short comparison with previous treatments is presented.

## 2. A toy model

### 2.1. General features and steady motion

This model is given by a massless and spinless field $\phi$ satisfying the wave equation

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}\right)\phi=0 \tag{2.1}$$

with the boundary conditions

$$\phi(t,0)=\phi(t,l)=0. \tag{2.2}$$

The Lagrangian, the canonically conjugate momentum and the Hamiltonian are respectively:

$$\begin{aligned} L&=\tfrac{1}{2}\int_0^l [\dot\phi^2-(\partial_z\phi)^2]\,\mathrm{d}z \\ \varpi&=\frac{\delta L}{\delta\phi}=\dot\phi \\ H&=\tfrac{1}{2}\int_0^l [\varpi^2+(\partial_z\phi)^2]\,\mathrm{d}z. \end{aligned} \tag{2.3}$$

The standard quantization condition is

$$[\phi(t,z),\dot\phi(t,z')]=\mathrm{i}\delta(z-z'). \tag{2.4}$$

The task now is to study the problem on the segment $[0,l]$ by considering $l$ a time-dependent variable. In this way the field variables acquire a new time dependence through the boundary conditions (equation (2.2)) and the Hamiltonian acquires a further dependence through the integration limit.

The boundary conditions (equation (2.2)) suggest the representation

$$\phi=\sqrt{\frac{2}{l}}\sum_n q_n \sin \pi nz/l \qquad \varpi=\sqrt{\frac{2}{l}}\sum_n p_n \sin \pi nz/l \tag{2.5}$$

with the inversion formulae

$$\begin{aligned} q_n&=\sqrt{\frac{2}{l}}\int_0^l \phi(z)\sin \pi nz/l\,\mathrm{d}z \\ p_n&=\sqrt{\frac{2}{l}}\int_0^l \varpi(z)\sin \pi nz/l\,\mathrm{d}z. \end{aligned} \tag{2.5$'$}$$

The relations (2.5) and (2.5′) allow us to calculate the explicit time variation of the mode operators $q_n$ and $p_n$.

$$\dot q_n=\dot l\frac{\partial q_n}{\partial l} \qquad \dot p_n=\dot l\frac{\partial p_n}{\partial l}. \tag{2.6}$$

Taking into account the different dependencies on $l$ we write

$$\frac{\partial q_n}{\partial l}=-\frac{1}{2l}q_n-\frac{\pi n}{l^2}\sqrt{\frac{2}{l}}\int_0^l z\phi(z)\cos \pi nz/l\,\mathrm{d}z+\sqrt{\frac{2}{l}}\phi(l)\sin \pi n.$$

The third term vanishes, precisely because of the boundary conditions, and some calculations, quite standard if lengthy, allow us to obtain the expression

$$\frac{\partial q_n}{\partial l}=\frac{1}{l}\sum_{m\neq n}\frac{2mn}{m^2-n^2}(-1)^{m+n}q_m \tag{2.7}$$

together with the completely analogous one for $p$.

In the mode representation we have for the Hamiltonian the representation

$$H=\tfrac{1}{2}\sum_n [p_n^2+(\pi n/l)^2 q_n^2] \tag{2.8}$$

and for the derivative the expression

$$\frac{\partial H}{\partial l}=\sum_n\left[p_n\frac{\partial p_n}{\partial l}+\omega_n^2 q_n\frac{\partial q_n}{\partial l}\right]-\frac{2}{l}\sum_n\omega_n^2\frac{q_n^2}{2}\qquad \omega_n=\frac{\pi n}{l}$$

which is reduced to the very simple form

$$\frac{\partial H}{\partial l}=-\frac{1}{l}\left[\sum_n(-1)^n\omega_n q_n\right]^2. \tag{2.9}$$

With the introduction of the absorption and emission operators

$$q_n=\frac{\mathrm{i}}{\sqrt{2\omega_n}}(c_n-c_n^\dagger)\qquad p_n=\sqrt{\frac{\omega_n}{2}}(c_n+c_n^\dagger).$$

It also results

$$H=\sum_n\omega_n(c_n^\dagger c_n+\tfrac{1}{2})\qquad \frac{\partial H}{\partial l}=\frac{1}{2l}\left[\sum_n(-1)^n\sqrt{\omega_n}(c_n-c_n^\dagger)\right]^2. \tag{2.10}$$

The operator $c^\dagger$ creates states whose energy is time dependent; this is the very idea of the adiabatic treatment [2]: the states follow the external parameters in their evolution but transitions are induced if the evolution is not infinitely slow.

According to the usual formalism for the adiabatic approximation in the case of discrete spectra, we consider a state $|\Psi\rangle$ evolving with the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\Psi\rangle=H|\Psi\rangle$$

and a set of 'adiabatic' eigenstates

$$H|k\rangle=E_k(l)|k\rangle.$$

For the projection coefficient

$$\gamma_k(l)=\exp\left[\mathrm{i}\int_{t_0}^{t}E_k(l)\,\mathrm{d}\tau\right]\langle k|\Psi\rangle$$

we have the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_k(l)=\dot{l}\sum_{j\neq k}\mathrm{e}^{\mathrm{i}\Phi}\frac{1}{E_j-E_k}\left\langle k\left|\frac{\partial H}{\partial l}\right|j\right\rangle\gamma_j(l) \tag{2.11}$$

with

$$\Phi=\int_{t_0}^{t}(E_k-E_j)\,\mathrm{d}\tau. \tag{2.11$'$}$$

Taking as an initial condition at $t=0$, $l=l_0$ the vacuum state: $|\Psi\rangle=|0\rangle$ the only different state reached at first order is the two-particle state; the corresponding $\gamma_0$ coefficient is

$$\gamma_2(l)=\frac{1}{2^{3/2}}\int_{l_0}^{l}\mathrm{e}^{\mathrm{i}\Phi}\gamma_0(l')\,\mathrm{d}l'/l'\qquad \gamma_0(l_0)=1 \tag{2.12}$$

$$\Phi=\mathrm{i}\pi(n_1+n_2)\int_{l_0}^{l}\frac{1}{\dot{l}'}\frac{\mathrm{d}l'}{l'}. \tag{2.12$'$}$$

When the transition probability is also small the correction to $\gamma_0$ is small, so we take to this order $\gamma_0 = \text{constant} = 1$; if furthermore the speed of the external parameter is constant $\dot{l} = u$, one can calculate explicitly the expressions (2.12) and (2.12′) and the transition probability from the vacuum to the two-particle state is obtained:

$$|\gamma_2|^2 = \frac{u^2}{4\pi^2} \frac{n_1 n_2}{(n_1+n_2)^4} 2\left[1 - \cos\left[\frac{\pi}{u}(n_1+n_2)\ln\frac{l}{l_0}\right]\right]. \tag{2.13}$$

Equation (2.13) completes the investigation of the model, in the case of steady motion of one boundary; the factor $u^2$ in front means that for any realistic situation the transition probability will be very small, so the use of the first approximation is fully justified.

### 2.2. Vibrations and resonance

It may be interesting to investigate the field configuration where one of the boundaries is vibrating so that $l = a + b\cos\Omega t$. In this case the phase appearing in the adiabatic formulae (equations (2.11) and (2.11′)) takes on a more complicated aspect:

$$\begin{gathered} e^{i\Phi} = \exp\left[i\pi\frac{n_1+n_2}{s}t\right]\left[\frac{a+s+b\,e^{-i\Omega t}}{a+s+b\,e^{i\Omega t}}\right]^{(\pi/\Omega)((n_1+n_2)/s)} \\ s = (a^2+b^2)^{1/2}. \end{gathered} \tag{2.14}$$

The expression in brackets may be expanded in series and it takes on the general form

$$\sum_{r=-\infty}^{+\infty} g_r\, e^{ir\Omega t}. \tag{2.15}$$

In order to calculate $\gamma$ one must integrate the expression, remembering that $1/l'$ also gives rise to an expansion like equation (2.15); the integrand is still made up of periodic functions but for the case $\pi(n_1+n_2)/s = \Omega r$, in this situation in fact it results

$$e^{i\Phi} = e^{i\Omega t}\left[\frac{a+s+b\,e^{-i\Omega t}}{a+s+b\,e^{i\Omega t}}\right]^r \tag{2.16}$$

so that in the integrand there appear some terms which are constant in $t$. These terms finally give a contribution which grows linearly with the total time $t_f - t_i$. It is clear that for too large $t_f - t_i$ the whole treatment cannot be correct; it is, however, true that there is the signal of a resonance where the transition is strongly enhanced. For every mode $n$ the corresponding frequency varies between $\pi n/(a+b)$ and $\pi n/(a-b)$, so the relevant quantity to define the resonance condition appears to be the geometrical mean of the extreme frequencies.

## 3. The real case

In the real case, as anticipated in the introduction, we consider a parallel plane capacitor: the distance between the plates (to be varied) will be $l$ and the plates will be two squares of side $\Lambda$, in every case $\Lambda \gg l$. One must get rid of the unphysical degrees of freedom of the EM field by a suitable choice of gauge. The most usual Coulomb gauge does not fit very well because if we vary $l$, at fixed $\Lambda$, the wavevectors allowed will vary in direction, which would result in momentum space in a time-varying gauge condition. For this reason, calling $z$ the direction orthogonal to the plates, the axial gauge $A_z = 0$, which is unaffected by variations of $l$, will be imposed [3].

Some notational conventions are used: the indices $i, j$ run from 1 to 3, the indices $a, b$ run from 1 to 2:

$$\varepsilon_{ab} \equiv \varepsilon_{ab3} \qquad v_T = (v_a v_a)^{1/2}.$$

The equations of motion for the vector and scalar potential are

$$(\partial_t^2 - \Delta)A_b - \partial_b(\partial_t\varphi - \partial_a A_a) = 0 \tag{3.1}$$

$$\Delta\varphi = \partial_a \dot{A}_a. \tag{3.2}$$

From the Lagrangian

$$L = \tfrac{1}{2}\int (E^2 - B^2)\,\mathrm{d}^3 r$$

$$= \tfrac{1}{2}\int [(\dot{A}_b - \partial_b\varphi)(\dot{A}_b - \partial_b\varphi) + (\partial_z\varphi)^2 - B^2]\,\mathrm{d}^3 r$$

the conjugate momenta are derived:

$$\Pi_b = \delta L/\delta \dot{A}_b = (\dot{A}_b - \partial_b\varphi) = -E_b \tag{3.2}$$

whereas

$$E_z = \partial_z\varphi. \tag{3.3'}$$

In terms of the conjugate momenta equation (3.2) simplifies a great deal, reducing to

$$\partial_z^2\varphi = \partial_a\Pi_a \tag{3.2'}$$

which can be solved in standard way:

$$\varphi(z, r_a) = \int G(z, z')\partial_b\Pi_b(z', r_a)\,\mathrm{d}z' \tag{3.4}$$

where $G$ is the Green function of $\partial_z^2$ with the correct boundary conditions for the problem.

The Hamiltonian of the system is

$$H = \tfrac{1}{2}\int \mathrm{d}^2 r_\perp \left\{\int \mathrm{d}z(\Pi_\perp^2(z) + B^2(z)) - \int \mathrm{d}z\,\mathrm{d}z'\,\partial_a\Pi_a(z)G(z, z')\partial_b\Pi_b(z')\right\} \tag{3.5}$$

with the magnetic field given by

$$B_z = \varepsilon_{ab}\partial_a A_b \tag{3.6}$$

$$B_a = -\varepsilon_{ab}\partial_z A_b. \tag{3.6'}$$

The general boundary conditions are

$$\begin{aligned} \Pi_a(0, r_b) &= \Pi_a(l, r_b) = 0 \\ B_z(0, r_b) &= B_z(l, r_b) = 0 \end{aligned} \tag{3.7}$$

but asking only for oscillating modes [4] we require 'cosine' conditions for $E_z$ and $B_a$ which also imply

$$\begin{aligned} &A_b(0, r_a) = A_b(l, r_a) = 0 \\ &\varphi(0, r_a) = \varphi(l, r_a) = 0 \end{aligned} \tag{3.7'}$$

and in this way we get the explicit form for $G$

$$G(z, z') = -\frac{2l}{\pi^2}\sum_n \frac{1}{n^2} \sin \pi nz/l \sin \pi nz'/l \tag{3.8}$$

since with this form $\partial_z^2 G$ acts as unity for the functions vanishing at the boundaries.

We expand the potential $A$ and the conjugate momentum $\Pi$. Since we are interested in the dynamics along the $z$-axis we take the usual plane-wave expansion in the $x$ and $y$-directions.

$$\begin{aligned} A_b(z, r_a) &= \frac{1}{\Lambda}\sqrt{\frac{2}{l}}\sum_p \mathrm{e}^{\mathrm{i}pr}\sum_n Q_b^{(n)}(p) \sin \pi nz/l \\ \Pi_b(z, r_a) &= \frac{1}{\Lambda}\sqrt{\frac{2}{l}}\sum_p \mathrm{e}^{\mathrm{i}pr}\sum_n P_b^{(n)}(p) \sin \pi nz/l. \end{aligned} \tag{3.9}$$

With the standard relations

$$\begin{aligned} &Q^*(p) = Q(-p) \qquad\qquad P^*(p) = P(-p) \\ &[Q_a^{(n)}(p), P_b^{(m)*}(p')] = \mathrm{i}\delta_{ab}\delta_{mn}\delta_{p,p'}. \end{aligned} \tag{3.10}$$

With the same procedures used in the previous section it is possible to calculate the derivative of the mode operators:

$$\frac{\partial Q^{(n)}}{\partial l} = \frac{1}{l}\sum_{m\neq n}\frac{2mn}{m^2 - n^2}(-1)^{m+n}Q^{(m)} \tag{3.11}$$

the same for $P$.

It is understood that $P$, $Q$ and $p$ are 2D vectors. A simplification is obtained by introducing for every mode $p_a$ the tangent unit vector $\tau_a = p_a/p_\perp$ and the normal $\nu_a = \varepsilon_{ab}\tau_b$ and the corresponding components of $Q$ and $P$: $Q_\tau = Q_a\tau_a$, and so on; note that $\tau$ and $\nu$ do not depend on $l$ so this operation commutes with the $l$-derivative.

We may now collect all the results and give the form of the Hamiltonian and its derivative in terms of the mode operators†:

$$\begin{aligned} H = \tfrac{1}{2}\sum_n\sum_p \{&|P_\nu^{(n)}(p)|^2 + [1 + (pl/\pi n)^2]|P_\tau^{(n)}(p)|^2 \\ &+ [p^2 + (\pi n/l)^2]|Q_\nu^{(n)}(p)|^2 + (\pi n/l)^2|Q_\tau^{(n)}(p)|^2\} \end{aligned} \tag{3.12}$$

$$\frac{\partial H}{\partial l} = \frac{1}{l}\sum_p\left\{\frac{p^2l^2}{\pi^2}\left|\sum_n \frac{(-1)^n}{n}P_\tau^{(n)}(p)\right|^2 - \frac{\pi^2}{l^2}\sum_\alpha\left|\sum_n\frac{(-1)^n}{n}Q_\alpha^{(n)}(p)\right|^2\right\} \qquad \alpha = \tau, \nu. \tag{3.13}$$

It is now convenient to introduce the energy of the mode

$$\omega_{p,n}^2 = p^2 + (\pi n/l)^2$$

† Needless to say, there is an intrinsic ultraviolet cut-off in the phenomenon because the condition of reflectivity for the boundaries cannot hold for very high frequencies.

and an $l$-dependent canonical transformation:

$$Q_{\tau}^{(n)} = \frac{\omega l}{\pi n} Q_{\mu}^{(n)} \qquad P_{\tau}^{(n)} = \frac{\pi n}{\omega l} P_{\mu}^{(n)}$$

which implements the transition from the axial gauge to the Coulomb gauge and brings the Hamiltonian to the standard form

$$H = \tfrac{1}{2} \sum_{p,n} \sum_{\alpha} [\omega^2 |Q_{\alpha}^{(n)}(p)|^2 + |P_{\alpha}^{(n)}(p)|^2] \qquad \alpha = \mu, \nu. \tag{3.14}$$

It also gives:

$$\frac{\partial H}{\partial l} = \frac{1}{l} \sum_{p} \left\{ \left| \sum_{n} \frac{(-1)^n p}{\omega_{p,n}} P_{\mu}^{(n)}(p) \right|^2 - \left| \sum_{n} (-1)^n \omega_{p,n} Q_{\mu}^{(n)}(p) \right|^2 - \left| \sum_{n} (-1)^n \frac{\pi n}{l} Q_{\nu}^{(n)}(p) \right|^2 \right\}. \tag{3.15}$$

The introduction of the usual emission and absorption operators

$$c_{\alpha}^{(n)}(p) = \frac{1}{\sqrt{2\omega_{p,n}}} [P_{\alpha}^{(n)}(p) - \mathrm{i}\omega_{p,n} Q_{\alpha}^{(n)}(p)]$$

$$c_{\alpha}^{(n)\dagger}(p) = \frac{1}{\sqrt{2\omega_{p,n}}} [P_{\alpha}^{(n)\dagger}(p) + \mathrm{i}\omega_{p,n} Q_{\alpha}^{(n)\dagger}(p)]$$

makes explicit the action of the derivative of the Hamiltonian $\partial H/\partial l$ on the Fock states. In particular for the transition from the vacuum to the two-photon state we get

$$\langle n_1 p, n_2 - p | \partial H/\partial l | 0 \rangle = (2l)^{-1} [\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3]$$

$$\mathcal{T}_1 = 2(-1)^{n_1+n_2} p^2 (\omega_1 \omega_2)^{-1/2} \tag{3.16}$$

$$\mathcal{T}_2 = 2(-1)^{n_1+n_2} (\omega_1 \omega_2)^{1/2} \tag{3.16'}$$

$$\mathcal{T}_3 = 2(-1)^{n_1+n_2} n_1 n_2 (\pi/l)^2 (\omega_1 \omega_2)^{-1/2}. \tag{3.16''}$$

The first two terms are obtained for the $\mu$-polarization, the third for the $\nu$-polarization.

This shows that the dynamics can be factorized into the different transverse modes, provided we keep the modes $p$ and $-p$ together, which is clearly required by the conservation of momentum in the plane $x$–$y$.

Now we calculate the transition amplitude from the vacuum to two-photon state, according to the adiabatic approximation.

The projection coefficient from the initial vacuum to final two-photon state is obtained in the same way as in the previous section (see equations (2.11)–(2.13)):

$$\gamma_{\alpha} = \int_{l_0}^{l_f} \mathrm{d}l\, \dot{l} F_{\alpha}(l) \exp\left[ \mathrm{i} \int_{l_0}^{l} (\omega_1 + \omega_2) \frac{\mathrm{d}l'}{\dot{l}'} \right] \tag{3.17}$$

where we have

$$F_{\nu} = \frac{1}{l} (-1)^{n_1+n_2} \frac{1}{\omega_1 + \omega_2} \frac{\pi^2 n_1 n_2}{l^2 \sqrt{\omega_1 \omega_2}}$$

$$F_{\mu} = \frac{1}{l} (-1)^{n_1+n_2} \frac{1}{\omega_1 + \omega_2} \frac{\omega_1 \omega_2 + p^2}{\sqrt{\omega_1 \omega_2}}. \tag{3.18}$$

From now on we assume $\dot{l}=u=\text{constant}\ll 1$, which is certainly true for every macroscopic motion; then the phase in (3.17) is very large, and through an application of the Riemann-Lebesgue lemma (see appendix) we can write

$$\gamma_\alpha \simeq -\mathrm{i}u\left\{(\omega_1+\omega_2)^{-1}F_\alpha\bigg|_{l_f}\exp\left[\frac{\mathrm{i}}{u}\int_{l_0}^{l_f}(\omega_1+\omega_2)\,\mathrm{d}l\right]-(\omega_1+\omega_2)^{-1}F_\alpha\bigg|_{l_0}\right\} \tag{3.17'}$$

so that finally the transition probability is written as

$$|\gamma_\alpha|^2=u^2\{(\omega_1+\omega_2)^{-2}F_\alpha^2|_{l_f}+(\omega_1+\omega_2)^{-2}F_\alpha^2|_{l_0}-2(\omega_1+\omega_2)^{-1}F_\alpha|_{l_f}\}$$
$$\times(\omega_1+\omega_2)^{-1}F_\alpha|_{l_0}\cos\frac{\mathrm{i}}{u}\int_{l_0}^{l_f}(\omega_1+\omega_2)\,\mathrm{d}l. \tag{3.19}$$

Through the usual quantization condition $p_a=(2\pi/\Lambda)m_a$ we can connect this expression to a photon density. In fact it results that the number of photons of longitudinal wavenumber $n$ per unit of transverse momentum squared and unit of transverse area is

$$\frac{1}{\Lambda^2}\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}p^2}=\frac{1}{4\pi}\sum_\alpha|\gamma_\alpha|^2.$$

The cosine term oscillates very rapidly around zero as a function, for example, of $l_f$ so we tentatively drop it with respect to the other terms. In so doing we get, summing over $\alpha$,

$$\frac{1}{\Lambda^2}\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}p^2}=\frac{u^2}{4\pi}\frac{1}{l^2}\left\{\frac{1}{(\omega_1+\omega_2)^4}\frac{1}{\omega_1\omega_2}[2(\omega_1\omega_2)^2-p^2(\omega_1-\omega_2)^2+2p^2]|_f+\ldots|_0\right\}.$$

In the case where $p\ll\omega$ we obtain a very relevant simplification, because the system behaves as if it has only one dimension, and the expression in equation (3.19) in fact becomes

$$|\gamma|^2\simeq u^2\frac{4}{\pi^2}\frac{n_1n_2}{(n_1+n_2)^4}.$$

If we tried to sum over the longitudinal quantum numbers we would get a diverging expression like

$$u^2\frac{2}{3\pi^2}\sum_s\left(\frac{1}{s}-\frac{1}{s^3}\right) \tag{3.20}$$

but, as has already mentioned, there is always an ultraviolet cut-off.

The above expressions require $l_f$ to remain different from $l_0$. The correct zero result for $l_f\to l_0$ is reproduced in equation (3.19), where the oscillating term has not been dropped. The problem of finding the number of photons produced in the process of mutual motion of two plane parallel plates is solved by equation (3.19); the actual number is very small because we always have a term $u^2$ in front, which for every macroscopic system is very small; this property makes all approximations well justified but, unfortunately, it also makes every experimental investigation very difficult.

Since the phase related to the adiabatic treatment in this more realistic case is more complicated, a discussion of the possible resonance conditions is not feasible in detailed analytical form. It appears, however, very likely that conditions of such a kind may exist; it is also clear that these conditions will unavoidably also depend, in particular if $p$ is not negligible with respect to $\omega$, on the transverse dimension $\Lambda$ whose role has essentially been ignored in all the previous discussion.

## 4. Conclusions and comparison with other treatments

Since problems more or less strictly related to that studied in this paper have been repeatedly considered it is necessary to present a comparison with the previous treatments. In the present paper the existence of two boundaries in relative motion is essential, so the comparison with situations where there is only one boundary is not straightforward; in those cases, in fact, it is evident that by Lorentz invariance, only the acceleration may possibly give rise to emission of quanta [5, 6]. Moreover, looking at equation (2.13), one sees that when $l_f \to \infty$ at fixed $l_f - l_0$ the transition probabilities calculated with constant speed of the boundary go to zero as they must.

A paper where the system is very similar to the present one has been written by Castagnino and Ferraro [7], continuing a line of investigation initiated by Moore [8]. The way they deal with the problem is quite different, but given the same physical starting-point the results are comparable; in particular their expression for the total number of particles looks like equation (3.20) of the present paper and the same can be said of the analogous result of [8]. The physical situation is the same because here the vacuum is set sharply at the initial time $t_i$ and the state is observed sharply again at $t_f$; in this sense one can speak of infinite accelerations. The logarithmic divergence is here considered unphysical due to the ultraviolet cut-off originating from the finite reflectivity of every physical surface.

As already stated, in the present paper and in the quoted references, one foresees a tiny photon emission because of the smallness of the coefficient $u^2$ or in every case of the macroscopic speeds; for such a reason the possibility of producing resonance conditions might be interesting because then the number of the emitted photons should increase with time, at least as long as the approximate calculations are trustworthy. In this context one may note that when $p$ is not negligible but nevertheless small with respect to $\omega$ a sort of 'non-relativistic' expansion of the type $\omega_{p,n} = \pi n/l + p^2 l/2\pi n$ could make the analytical study of the resonance condition complicated but not hopeless.

## Appendix

For completeness the connection between equations (3.17) and (3.17′) is shown here.

If we have

$$\mathcal{T} = \int_{x_i}^{x_f} \mathrm{d}x\, F(x) \exp \frac{\mathrm{i}}{u} \int_{x_i}^{x} w(y)\, \mathrm{d}y$$

with $w$ definite positive and $u \to 0$, we define

$$\alpha(x) = \int_{x_i}^{x} w(y)\, \mathrm{d}y$$

which, being monotonically increasing, can be inverted as

$$x = x(\alpha) \qquad x_i = x(0)$$

$$\mathcal{T} = \int_0^{\alpha_f} F(x(\alpha))\, \mathrm{e}^{\mathrm{i}\alpha/u} \frac{\mathrm{d}x(\alpha)}{\mathrm{d}\alpha}\, \mathrm{d}\alpha$$

if $\Phi(\alpha) \equiv F(x(\alpha))(\mathrm{d}x/\mathrm{d}\alpha)$ we also have

$$\mathcal{T} = \int_0^{\alpha_f} \Phi(\alpha)\, \mathrm{e}^{\mathrm{i}\alpha/u}\, \mathrm{d}\alpha = [-\mathrm{i}u\Phi(\alpha)\, \mathrm{e}^{\mathrm{i}\alpha/u}]_0^{\alpha_f} + \mathrm{i}u \int \frac{\mathrm{d}\Phi}{\mathrm{d}\alpha}\, \mathrm{e}^{\mathrm{i}\alpha/u}\, \mathrm{d}\alpha.$$

The second term is $O(u^2)$ and reverting to the original variables

$$\mathcal{T} \simeq -\mathrm{i}u \left\{ \frac{F(x_f)}{w(x_f)} \exp \frac{\mathrm{i}}{u} \int_{x_i}^{x_f} w(y)\, \mathrm{d}y - \frac{F(x_i)}{w(x_i)} \right\} + \mathrm{O}(u^2).$$

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